Mr Kepler's shapes

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UNIT 15

In this unit, the students engage in a range of activities with shape whilst considering how geometry can contribute to our appreciation of the universe. They also learn about the mathematician Kepler and his work.

Mathematical content

Shape and Space (AT3)
Describing and constructing symmetrical shapes
Nets and polyhedra
Constructing curves and loci

Using this unit

The unit is intended to be accessible to students at all tiers of GCSE.

The unit is expected to take about 3 hours. There are three sections in the unit. Each section could be done in a long lesson, with scope for homework as well. The sections do not all have to be done at the same time but, if they are done at different times, reference will need to be made to earlier work. Each one relates to a different aspect of Kepler's work and contain descriptions of his work and some quotations from his writings. There are core and extension activities for different elements of shape and space work. It is recommended that each section begins with a whole class discussion/thinking activity. Each section offers opportunity for student reflection.

There are practical elements to the work so students will need access to drawing equipment and suitable paper/card for the polyhedra work. Templates for squares, equilateral triangles and regular pentagons would be useful. Already constructed regular solids would also be helpful.

Spiritual and moral development

The aim of the unit is to promote a sense of wonder for the students about the Universe in which they live. It also aims to develop their awareness of how religious belief can both motivate and stimulate the work of mathematicians and scientists.

T E A C H E R ' S N O T E S

Background

Kepler is best known for his work on planetary motion. Less often mentioned is the passion he had to make known the glory of God's works in creation. His faith in God convinced him that there must be a harmony in the universe. This in turn convinced him of the mathematical design of the Universe and that explanations of the world needed a thorough grounding in geometry. Nevertheless, he was always keen to check his theories against observations and he also sought plausible physical causes.

In *The Six-Cornered Snowflake*, Kepler sought to explain the hexagonal symmetry through geometry. He saw a fundamental connection between the formation of a snowflake in the atmosphere, the constructions of a honeycomb by a family of bees and the growth of a pomegranate. According to his theory, the regular, symmetrical patterns that arise in each case can be described and explained in terms of "space-filling geometric figures".

In both *The Mystery of the Cosmos* and *The New Astronomy*, he used his knowledge of Greek geometry as sources of ideas for new theories. Euclid's *Elements* and Appolonius' *Conics* were highly regarded textbooks in Kepler's day and these would have introduced him to the regular (Platonic) solids and to the ellipse as a conic section, respectively.

Additional sources

Morris Kline, *Mathematics in Western Culture* (Penguin, 1990)
Ian Stewart and Martin Golubitsky, *Fearful Symmetry – Is God a Geometer?* (Penguin 1992)
John Fauvel and Jeremy Gray (eds), *The History of Mathematics – A Reader* (Open University, 1987)
Keith Devlin, *Mathematics – The Science of Patterns* (Scientific American Library, 1994)
Istvan Hargittai and Magdolna Hargittai, *Symmetry – A Unifying Concept* (Shelter Publications, 1994)

Polydron, Clixi or ATM Mats all offer alternative ways of exploring nets and polyhedra.

Notes on the activities

In each of the three sections, there is a quotation from Kepler's writing followed by some questions which are there to encourage the students to reflect. In order for this to be effective:

- students must be given time to focus upon the questions on their own;
- students must be encouraged to make a response, either through writing or contributing to class discussion;
- u students' views must be respected by the class;
- u the end of lessons could be given over to this kind of reflective period.

The Six-cornered Snowflake

Start this section by putting up an acetate of the snowflakes (if OHP is not available, then worksheets could be copied). Give them time to appreciate the images.

Ask students to say what they notice about the snowflakes. Get them to describe any patterns, shapes or symmetry that they see. As necessary, make sure they are clear that all snowflakes have hexagonal symmetry. Following this the snowflake sheet can be issued. You may wish to demonstrate the snowflake construction to the whole class.
A possible extension task not suggested on the sheet is to explore creating snowflake images using LOGO.

The Mystery of the Cosmos

Start this section by asking students to try to imagine solid shapes for which all the faces are identical. Ask individual students to describe or draw their solids for the rest of the class. (This could then extend into a practical investigation if time were available.) Alternatively, the sheet could be issued and the five solids discussed.

You will need to decide how you want the students to produce the nets in question 2. This will depend on available resources and the extent to which you wish drawing skills to be developed. Question 4 is really an extension task and requires some quite high level reasoning.

The New Astronomy

Start this section by asking students to imagine what shapes you get if you slice a cylinder in different ways. It will help if you have some way of demonstrating the slicing process. Make sure that they see that the cross-section can be either a circle or an ellipse. (A possible extension to this is to use a double cone and include also the parabola and the hyperbola.)

Some students may need a demonstration of some of the ellipse construction methods. The methods in question 1 and 2 can be extended by exploring what happens when you have the point P in different places. Do you still get an ellipse? Is it the same?

Answers

Task 1:

1. a) rotational symmetry of order 3; rotational and reflection symmetry of order 6; rotational and reflection symmetry of order 4; rotational and reflection symmetry of order 6.
   b) Shape 2 and Shape 4 could be snowflakes because of their rotational and reflection symmetry of order 6.

4. a) Use the same method as before but start with 6, 8, 10 and 16 sectors respectively.

Task 2

1.

Name	Number of faces	Number of edges	Number of vertices
Tetrahedron	4	6	4
Cube	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

3. a) $F + V = E + 2$.

4. Students can be encouraged to explore the different ways of arranging regular shapes in a net so that at least three edges join to make a vertex. Once a hexagon or above is considered the interior angles become too large. For pentagons and squares the maximum possible number of edges at a vertex is 3 (dodecagon and cube). For triangles 3, 4 and 5 edges can meet at a vertex (tetrahedron, octahedron, icosahedron) but not 6. This is therefore an exhaustive list of the possible regular solids.

Mr Kepler’s shapes

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UNIT 15

In this unit, you will be doing a variety of shape activities, but you will also be learning about an important mathematician, Johannes Kepler, and what he understood about the Universe.

The six-cornered snowflake

Johannes Kepler lived from 1571 to 1630 in different places in Germany, Austria and the Czech Republic. He was a mathematician, astronomer and scientist. He went to University in Tubingen when he was 16. It was said of him there that he had “such a superior and magnificent mind that something special may be expected of him”.

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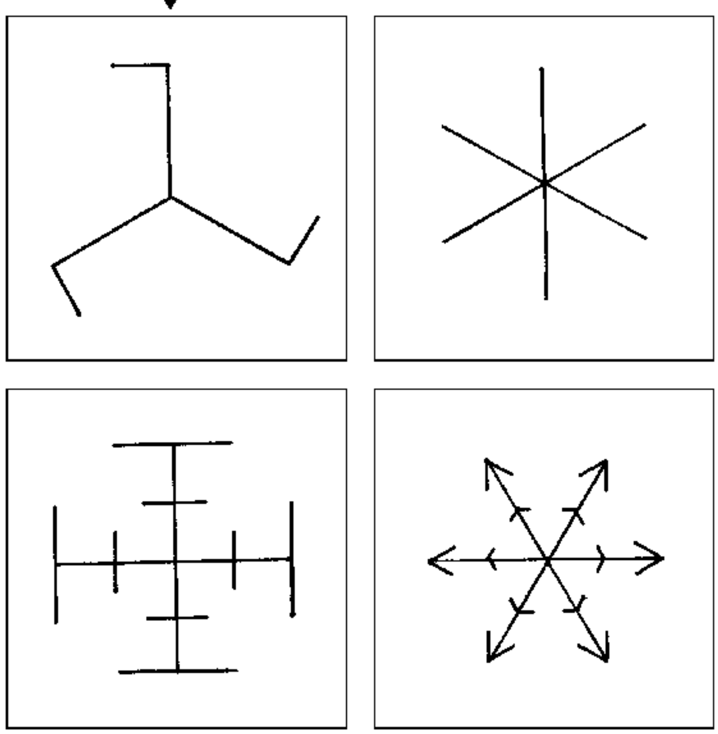
In about 1600, Johannes Kepler wrote a remarkable essay called the *Six-Cornered Snowflake*. It was written as a New Year’s present to Johann Matthaus Wacker von Wackenfels, the man who paid his salary. Kepler knew that all snowflakes had rotational symmetry and reflection symmetry both of order 6. He wondered why this was so. He thought very hard about it and, using his knowledge of shape, concluded that it was something to do with packing objects as closely together as possible.

1

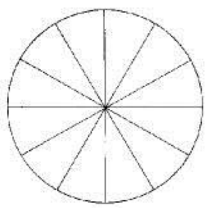
1. Look at the diagrams alongside.
   a) Describe the symmetry of each of the shapes
   b) Which of them could be diagrams of snowflakes? Explain how you know.



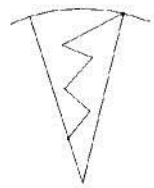
1

Task 1 continued

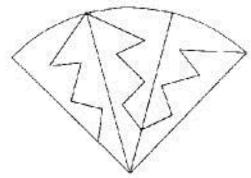
2. Follow the stages described below in order to produce a snowflake image.



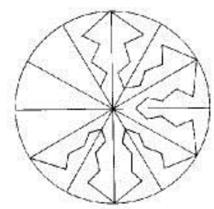
Stage 1:
On a piece of plain A4 paper, draw a circle of radius 8 cm and divide it into 12 equal sectors



Stage 2:
Choose two adjacent sector lines. Mark a point on one at the circumference and mark a point on the other at about 2 cm from the centre. Then join the two points with a line design like the one shown.



Stage 3:
Draw the reflection of the pattern in one of the sector lines. Then reflect this image in the next sector line.



Stage 4:
Continue this reflection process around the circle until the snowflake is produced. Then cut out the final shape.

3. You can produce more complex snowflakes by changing your line design. Try the method again, but this time use a more complicated line design.

4. Snowflakes have rotational symmetry of order 6.
   a) How would you adapt the method in question 2 to produce shapes with rotational symmetry of order 3, 4, 5 or 8?
   b) Use these adapted methods to produce some shapes with different orders of rotational symmetry.

I thank thee O Lord our Creator, that thou hast permitted me to look at the beauty in thy work of creation. I exalt thee in the works of thy hands.

Kepler in *Harmony of the World*

Kepler was fascinated enough by snowflakes to write an essay.

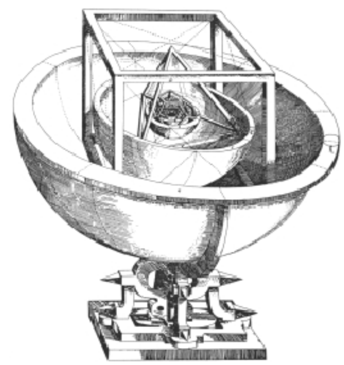
- How do you react to the beauty and complexity of snowflakes?
- Have you ever been amazed by the beauty of other things in the natural world?
- Do you ever take time to stop and look more closely at the natural world you see around you?
- Do you ever take time to wonder why things are like they are?

The mystery of the cosmos

In 1594, Kepler was encouraged to leave university and to take up the post of astronomy teacher and professor of mathematics and morals in Ganz, South Austria. It was about a year later when he had an amazing idea one Summer afternoon which led him to write and publish his book *The Mystery of the Cosmos* or *The Cosmographic Secret*.

At that time, there were only six known planets: Mercury; Venus; Earth; Mars; Jupiter and Saturn. Kepler wanted to explain why there were exactly six planets and why they were at particular distances from the Sun. He experimented with lots of different ideas until he arrived at a model that fitted with there being only 5 regular solids. In this model, the orbits of the planets were on the surface of spheres. Each solid was carefully chosen to work out accurately the size and position of the next sphere. Each time the solid nested perfectly inside the next sphere and the sphere perfectly inside the next solid.

Kepler's diagram shows his idea: the outer sphere (on which Saturn moves) contains a cube, inside which is a sphere for the orbit of Jupiter. In that sphere was a tetrahedron and the sphere inside that was for Mars' orbit. Next came the dodecahedron with a sphere inside for the Earth's orbit, followed by the icosahedron with the Venus sphere after that. Finally came the octahedron with the Mercury sphere inside this again. Had he solved the mystery of the Cosmos? Sadly, Kepler's very imaginative idea was wrong. The spheres were not quite right and we of course now know that there are at least nine planets.

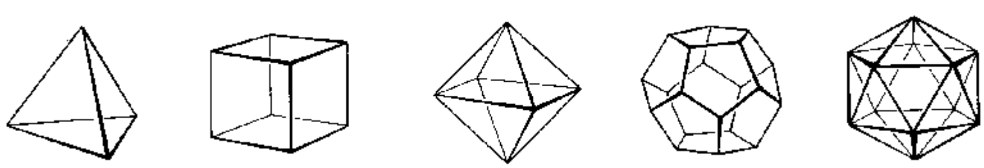


Kepler's planetary model

2

1. For each of the regular solids, complete the table.

Name	Number of faces	Number of edges	Number of vertices
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Icosahedron			



Regular solids

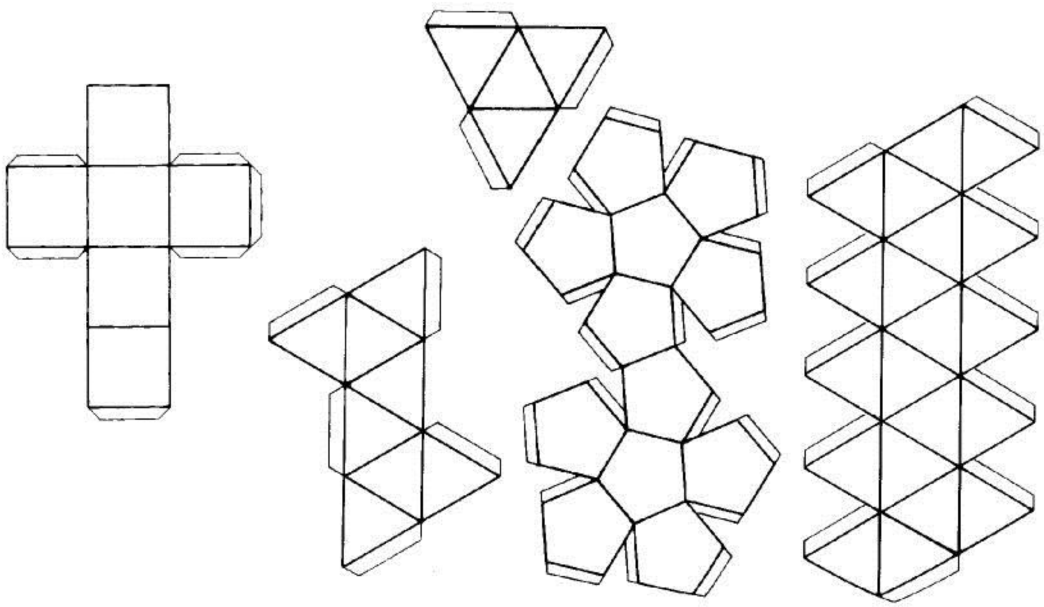
2

Task 2 continued

2. For each of the solids construct a net. Try folding the net to produce the solid. You may find it helpful to use a template to draw the net. Alternatively, your teacher may have equipment you can use for constructing solids.

3. There is a connection between the number of faces, edges and vertices of the regular solids.
   a) Try to work out the connection and write it as a formula. It is normally called Euler's formula.
   b) Investigate to see if the formula works for other solids as well.

4. Can you explain why there are no more than five regular solids?



Happy the man who devotes himself to the study of the heavens;
he learns to set less value on what the world admires most;
the works of God are for him above all else, and their study will furnish him
with the purest of enjoyments.
Kepler in the preface of *The Mystery of the Cosmos*

Kepler's response to the patterns he found in the heavens was to praise God.

- Have you ever spent time just gazing at the night sky?
- How do you respond to the amazing Universe in which we live?

The new astronomy

Although Kepler's idea in the *Mystery of the Cosmos* was not accepted, it did bring him into contact with Tyco Brahe, the leading astronomer of the day. By 1600, Kepler was working as an assistant to Brahe in Prague. Kepler's task was to observe the orbit of Mars. Brahe died in 1601 and Kepler took over as imperial mathematician to Emperor Rudolf II. He continued to examine the orbit of the planet Mars and in 1609 he published a book called *The New Astronomy*.

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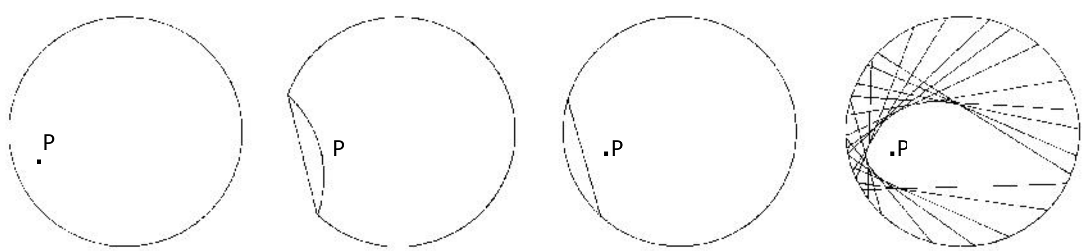
Kepler thought that there was great harmony in the world and so expected any explanations of the world to require geometry. At the time, people believed that the orbits of the planets had to be related to circles because circles were perfect shapes. However, this meant that they needed a very complicated system of circles with centres that moved in circles and so on. Kepler had the inspired idea to give up on the circle and consider the ellipse instead. This was a shape mathematicians had known about for almost 2000 years. By choosing the ellipse for an orbit, everything became very simple and Kepler was convinced that he had found a better theory.

3

1. Follow the stages below to construct an ellipse.



Stage 1: Cut out a circle of radius 8 cm and mark a point P quite close to the edge.

Stage 2: Fold the circle over so that the circumference just touches the point.

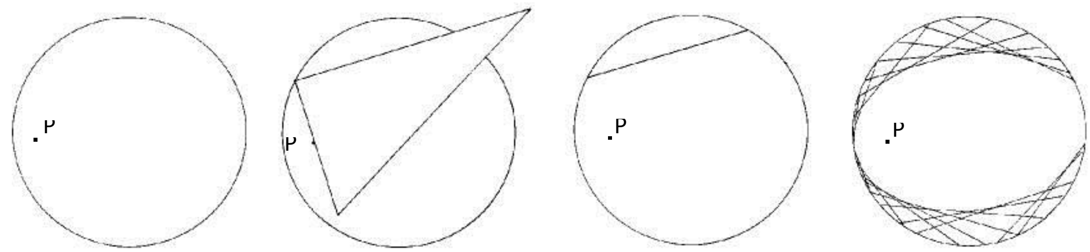
Stage 3: Unfold the circle and go over the crease in pencil.

Stage 4: Continue folding and drawing until the ellipse becomes clear.

3

Task 3 continued

2. Follow the stages for this alternative method for constructing an ellipse.



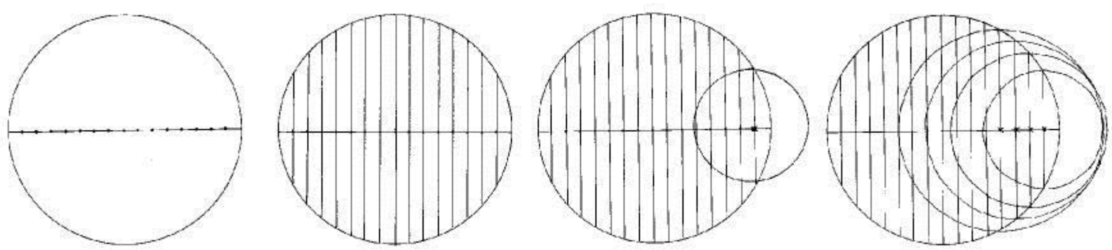
Stage 1:
Draw a circle of radius 4 cm and mark a point P near the edge.

Stage 2:
Place a set square so that one edge touches P and the right angle is on the circumference of the circle.

Stage 3:
Draw the chord of the circle made by the other side.

Stage 4:
Repeat stages 2 and 3 for different set square positions until the ellipse becomes clear

3. Follow the stages for a final method for constructing an ellipse.



Stage 1:
Draw a circle of radius 4 cm and mark off points every 5mm along a diameter.

Stage 2:
At each of the marked points draw chords perpendicular to the diameter.

Stage 3:
Choose one of the lines and treat it as a diameter of a circle. Draw that circle.

Stage 4:
Repeat the process for all of the perpendicular lines.

> The wisdom of the Lord is infinite, so also are His glory and His power ...
> To Him be praise, honour and glory throughout eternity.
> Kepler in *Harmony of the World*

Kepler believed that as he studied the world with its amazing harmony he would also discover more about God.

- Do you have a desire to know more about the world?
- If you do, explain what and why. If you do not, also try to explain why.
- What is it that motivates you to study?

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