The outer limits

UNIT 9

Sequences and series are investigated in order to develop the ideas of numbers being "as big as we like" and "as small as we like". This is illustrated by reference to the biblical examples of the numbers of stars in the sky and grains of sand on the seashore.

Using this unit

The early work in the unit deals with sequences and the summing of series. The distinction between divergent and convergent series is introduced without using the words "convergent" and "divergent".

Use of fractions is required and the ability to use a fractions calculator would be helpful.

Further work on very big numbers, using an example from the life of Abraham, is then brought in as a "bridge" from the idea of big numbers in mathematics to the immensity of the universe.

The paradox that results from treating infinity as if it were just another number is included and the unit concludes with a game which illustrates the density of the number line.

Students will need a calculator, preferably one with fractions facility.

Background

Mathematicians have sometimes had religious motivations for their study of mathematics. Georg Cantor (1845-1918) is an example of this in his study of the idea of infinity and "transfinite numbers". His contribution to human understanding of these ideas was massive and unique. He was a devout Lutheran Christian who was concerned to understand better what is meant by talk of God as being "infinite". He admitted that the study of mathematical infinity is beset by paradoxes but he believed that it is based on a reality beyond this universe which cannot be fully determined by any mathematical system.

Mathematical content

AT2
- **Generation and investigation of sequences and series**
- **Exploring patterns**
- **Investigation of concepts of infinity**

Spiritual and moral development

The aim of this unit is to promote a sense of wonder at how numbers can become very large and intervals on the number line very small. It also aims to create a sense of wonder at the immensity of the universe.

Notes on the Activities

Sequences

Task 1 answers:

1. 1, divide the previous term by 2
2. $^1/_{32}$, $^1/_{64}$, $^1/_{128}$

Task 3 answers:

1. 1.875, 1.9375, 1.96875
2. Tends to 2
3. Use of a spreadsheet may be helpful here.
   a) 5, b) 8, c) 11, d) Never

Task 5 answers:

1. 48
2. Divide by 4
3. The sum gets closer to 64

A proof of the divergence of the series
$1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$ is as follows:

First, consider the sum of two sequences:

$S_1 = a_1 + a_2 + a_3 + a_4 + \ldots$
$S_2 = b_1 + b_2 + b_3 + b_4 + \ldots$

and they are such that $b_1 \le a_1$, $b_2 \le a_2$, $b_3 \le a_3$, and so on, then $S_2 \le S_1$

Now consider
$S_1 = 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 \ldots$

We shall now construct a second sum S_2, in which each term is the same as, or less than, the corresponding term in S_1.

Replace $^1/_3$ by $^1/_4$

Then replace $^1/_5$ by $^1/_8$, $^1/_6$ by $^1/_8$

replace $^1/_7$ by $^1/_8$

Each term is being replaced by a smaller term.

Thus $S_2 = 1 + {}^1/_2 + {}^1/_4 + {}^1/_4 + {}^1/_8 + {}^1/_8 + {}^1/_8 + {}^1/_8 + \ldots$

Then we can write $S_2 = 1 + {}^1/_2 + {}^2/_4 + {}^4/_8 + \ldots$

i.e. $S_2 = 1 + {}^1/_2 + {}^1/_2 + {}^1/_2 + \ldots$

In other words we can make S_2 as large as we like. Since $b_1 \le a_1$, $b_2 \le a_2$, $b_3 \le a_3$, and so on, then $S_2 \le S_1$. Since we have shown that S_2 can be made as large as we like, and since S_1 is not smaller, then S_1 can have no limit.

Task 6 answers:

1. 1, 3, 9
2. 13
3. 29,524

Task 7 answers:

1. 1, 1.1, 1.21
2. 3.31
3. 15.93

As big or as small as you like

Task 8

1. Any attempt at this would be grossly inaccurate but it could initiate a useful discussion.

Task 9

The paradoxes may need some explanation to help the students to develop the idea of an infinite set.

Task 10

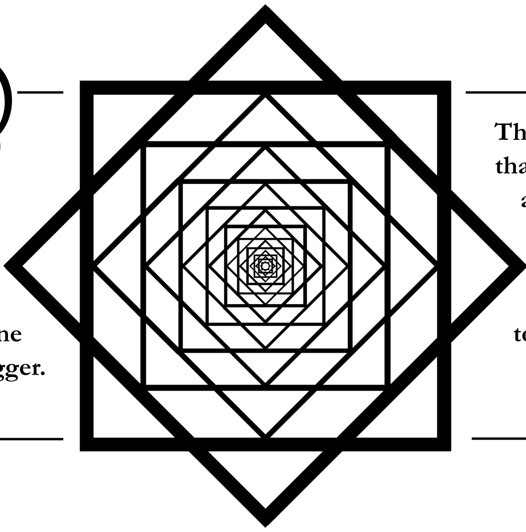
This game provides practice in the use of decimals. It may seem pointless to the students when they realise that the process can go on for ever and, indeed, that it should do so if numbers are always chosen within the intervals. But the point of the game is to help them to realise that it does so, i.e., that there is an infinite set of numbers between any two numbers on the number line.

The outer limits

UNIT 9

Think of a very big number.

Whatever number you think of, it is always possible for you to add one to it and make it even bigger.



This unit examines numbers that can get as big as we like and also numbers that can get as close to another number as we like. We will study them in order to get a better idea of what we mean by "infinity".

Sequences

1

Consider the sequence of numbers:

$1,\ ^{1}/_{2},\ ^{1}/_{4},\ ^{1}/_{8},\ ^{1}/_{16},\ \ldots$

Every sequence of numbers has a first number and a rule which tells us how we can work what the next number is and the one after that and so on.

1. What is the first number in the sequence and what is the rule for the sequence?
2. Use the rule to write down the next three terms after $^{1}/_{16}$.
3. Write down three more sequences with different first terms but which still obey the same rule.

The pattern in the sequence above should be clear - each term is half the one before, so each term is getting smaller.

2

What happens when you go as far as you like along the sequence?

3

Consider what happens when you make up a new sequence by adding up the terms of the sequence:

$1, {}^1/_2, {}^1/_4, {}^1/_8, {}^1/_{16}, \ldots$

We will call this a totaliser sequence. The first term is simply 1 and we shall write this as:

$T_1 = 1$

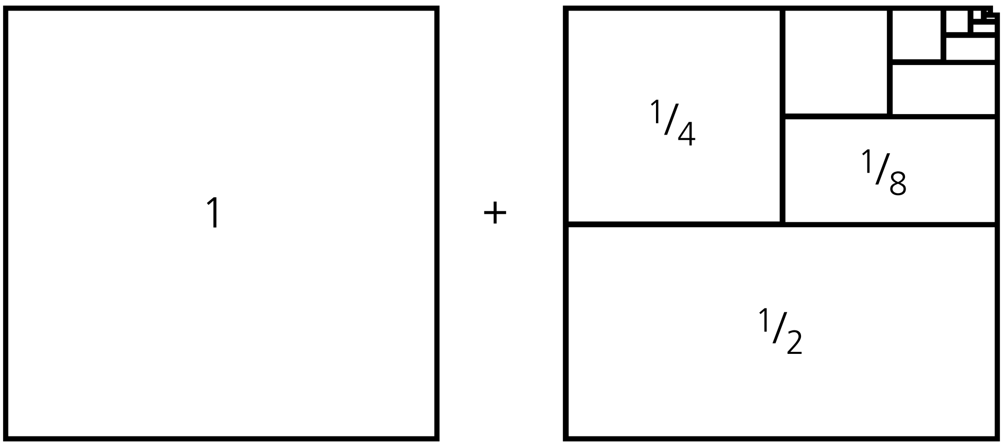
The second term in the totaliser sequence is obtained by adding 1 and ${}^1/_2$. We write this as:

$T_2 = 1 + {}^1/_2 = 1.5$

In the same way, $T_3 = 1 + {}^1/_2 + {}^1/_4 = 1.75$

1. Calculate the values of T_4, T_5, and T_6
2. What do you think happens as the totaliser sequence continues? Investigate.
3. a) At which term does the totaliser sequence pass 1.9?
   b) At which term does it pass 1.99?
   c) At which term does it pass 1.999?
   d) When does the totaliser sequence get to 2?

It seems as though you never get to 2, but you can get as close as you like by taking enough terms. Look at the pictures below:



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Explain in your own words how these pictures show that the totaliser sequence gets closer and closer to 2.

$1 + {}^1/_2 + {}^1\ {}_4 + {}^1/_8 + {}^1/_{16} + \ldots$
gets closer and closer to 2 without ever reaching it.
Mathematicians say that the limit of the sequence is 2.

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Now consider the sequence:

$$48,\ 12,\ 3,\ {}^{3}/_{4},\ {}^{3}/_{16},\ {}^{3}/_{64},\ \ldots$$

1. What is the first term?
2. What is the rule that produces the terms that follow?
3. What happens to the totaliser sequence in this case?

$$T_1 = 48,\ T_2 = 60,\ T_3 = 63,\ T_4 = \ldots$$

Now consider the sequence of numbers:

$$1,\ {}^{1}/_{2},\ {}^{1}/_{3},\ {}^{1}/_{4},\ {}^{1}/_{5},\ \ldots$$

Notice that the denominator of each term is one more than the denominator of the previous term.

Now consider what happens to the totaliser sequence in this case:

$$T_1 = 1,$$
$$T_2 = 1 + {}^{1}/_{2} = 1.5,$$
$$T_3 = 1 + {}^{1}/_{2} + {}^{1}/_{3} = 1.833\ldots$$

Can you guess a number which is the limit of this sequence?

6

Consider the following sequence:

First term: 1
Rule: multiply the term before by 3

1. Write down the first three terms.
2. Find the sum of the first three terms.
3. Use your calculator to add up the first ten terms.

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Now consider the sequence:

First term: 1
Rule: multiply the term before by 1.1

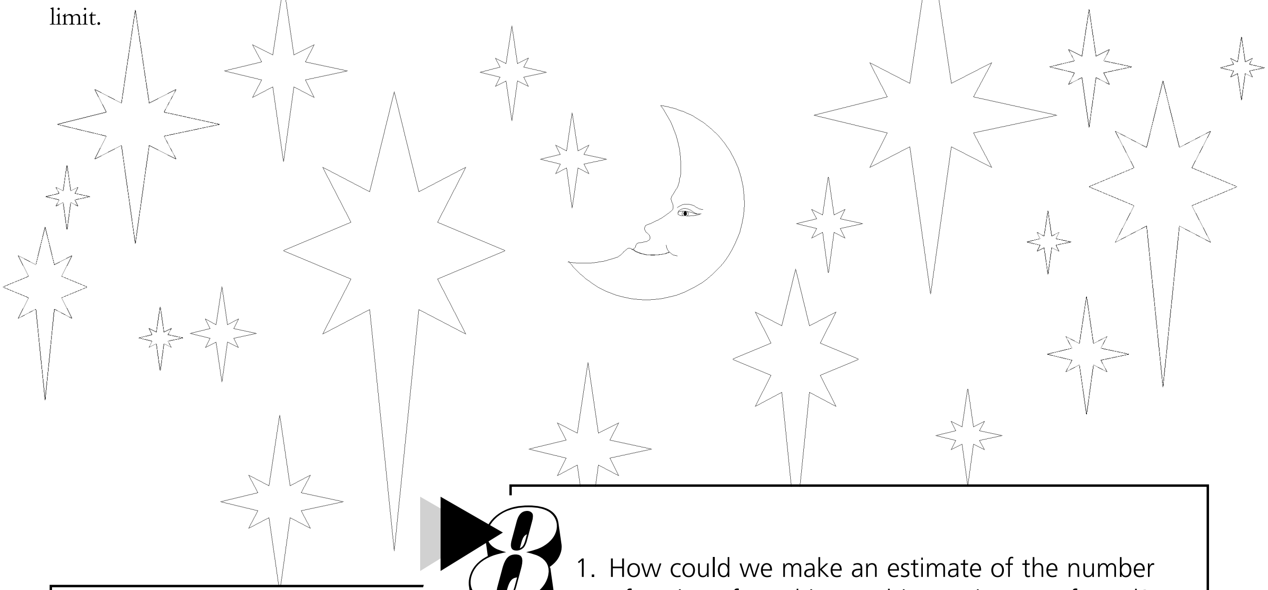
1. Write down the first three terms.
2. Find the sum of the first three terms.
3. Use your calculator to add up the first ten terms.

In all these sequences, the terms get bigger and bigger. In some cases, they move towards a limit but in others, e.g., the last two you have looked at, there is no limit.

As big or as small as you like

In your work on sequences, you have found that sums of terms sometimes get bigger and bigger without any limit. They become as big as we like because adding on terms makes them bigger without approaching a limit.

The sums of terms in other sequences get closer and closer to a limit. We can get as close to it as we like without actually reaching it. The gap becomes as small as we like.



In the Bible (in Genesis chapter 22 verse 17) it is recorded that God told Abraham that the number of his descendants would be as great as the number of grains of sand on the seashore and as great as the number of stars in the sky.

1. How could we make an estimate of the number of grains of sand in 1 cubic centimetre of sand?
2. Would that be good enough for us to make an estimate of the number of grains of sand in a cubic metre?
3. How many cubic metres of sand are there on your nearest beach?

In the days of Abraham there would have been no “light pollution” and no material pollution as we know it. As a result, the stars would have been much clearer in the sky. (People who have been at sea on a clear, moonless night often talk about the amazing sight of the Milky Way stretched across the sky.)

The idea of actually counting the stars would have been ridiculous to Abraham and the people of his time. Nowadays we know that there are actually more stars than people believed there to be years ago and we can be almost certain that there even more about which we do not know.

Whatever big number we think of, we can always make it bigger by adding to it, multiplying it by another number or squaring or cubing it. But even when we have done that, we are still a long way short of what people mean by saying that something is infinitely big.

Mathematicians have struggled for centuries to understand the idea of infinity. Some of them have done so because they want to understand what is meant by saying that God is infinite.

9

Here are two "paradoxes" to do with the idea of infinity.

1. Every whole number has a square

1	2	3	4	5
↓	↓	↓	↓	↓
1	4	9	16	25

There are therefore as many squares as there are whole numbers.

How many whole numbers are there which are not squares?

How many more numbers are there than there are squares?

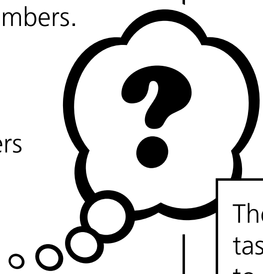
So how many whole numbers are there and how many square numbers are there?

2. Prime numbers form a "sub-set" of the set of positive whole numbers, that is, these numbers are all members of the set of positive whole numbers. Because some numbers are not prime, these are left behind when you take out the prime numbers.

How many prime numbers are there?

How many numbers are there when the prime numbers are taken out?

So how many more numbers are there than prime numbers?



A paradox is something that seems to contradict itself. From these examples it seems that adding two infinite sets together produces an infinite set and also that taking an infinite set away from an infinite set still leaves an infinite set.

The following task requires you to work in pairs:

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a) Choose one of you to start.
b) The one who starts chooses two numbers.
c) The second person chooses a number between them and then crosses out one of the boundary numbers.
d) The first now chooses a number in the interval that is left.
e) The exercise goes on until someone chooses a wrong number, i.e., one that is not in the interval.

Example: Alf and Bert play and Alf starts.

b) A chooses 1 and 2.
c) B chooses 1.7 and crosses out 2
d) A chooses 1.5 which he decides in within the interval from 1 to 1.7 that was given to him and crosses out 1.
e) Bert now has to choose a number in the interval 1.5 to 1.7 and he chooses 1.62 and so the game goes on.

"From time immemorial, the infinite has stirred emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully."
David Hilbert