Can you draw it?

UNIT 8

This unit deals with networks. The topic has been approached within a number of real situations including the design of micro-processor chips. This leads on to discussion of the anthropic principle.

Using this unit

The unit is designed to be used by students working towards Higher Level GCSE and will take from 2 to 3 hours. It requires no previous knowledge from other areas of mathematics.

It involves them in constructing various topological networks and investigating their traversability. Students are then helped to find Euler's Formula before going to work on planar graphs. They are then introduced to the anthropic principle.

Finally, the impossibility of K_5 is used to support an argument for a three-dimensional world.

Background

The anthropic principle states that the universe must have properties which allow the development of intelligent life. There are many ways in which our universe has such properties:

- It is expanding at exactly the right rate to allow galaxies and stars to form;
- It is sufficiently lumpy to allow matter to coalesce into galaxies, stars and planets;
- A particular isotope of carbon has a nucleus with precisely the right energy levels to allow its formation from simpler elements - and carbon is an essential component of life;
- The strength of the various forces in the universe - gravitational, electromagnetic and nuclear forces - are extremely precisely balanced; if they were at all different, the universe would be a very different place, and would almost certainly not support life. (continued over)

Mathematical content

AT1

- **Simple ideas of topology are used in an investigative mode to illustrate the nature of mathematical proof**

AT3

- **Some work on cross-sections of three-dimensional shapes**

Spiritual and moral development

The aim of the unit is to show students that some results, such as Euler's Formula, can be proved by logical argument. It also aims to present students with an argument that the universe is constructed in a way which specifically provides for the existence of creatures such as ourselves.

Background continued

Explanations for the anthropic principle depend on one's overall view of life, the universe and everything:

i) maybe we are just lucky that the universe is so hospitable;

ii) perhaps there are many alternative universes and life has arisen in this one because it is suitable;

iii) God designed a universe that would be precisely right for life because He wanted to create human beings with which He could have communion.

Further details of this fascinating subject can be found in Barrow and Tipler's monumental study, cited in Additional Sources.

Additional sources

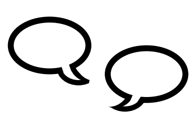
J. D. Barrow & F. J. Tipler, *The Anthropic Cosmological Principle* (OUP, 1986).
E. A. Abbott, *Flatland: A Romance of Many Dimensions* (Dover, 1992).
A. K. Dewdney, *The Armchair Universe* (W. H. Freeman, 1988).

Notes on the activities

The Königsberg Bridges

Students start by trying to find a path around various networks - the Königsberg Bridges' network and others which they construct for themselves. When thinking about what makes a network traversable or not, their attention can be directed to the number of odd nodes in the network. They will see that if there are zero or two odd nodes, then the network can be traversed - in the former case, starting at any point, in the latter, starting at one odd node and finishing at the other. This is because odd nodes must be end points of a journey, and a journey can only have two ends. On the other hand, even nodes can be both starting and finishing points, and so, if every node is even, every node can be the starting and finishing point for a traversal.

Euler's Formula



In the second section, students investigate the relationship between the number of arcs, nodes and regions in a planar graph. By compiling a table of results for their own networks, they can spot the pattern N + R = A + 2. They are then led through a proof. Alternatively, the proof could be presented through whole class discussion.

Planar graphs

Euler's result is used here to show the impossibility of drawing K_5 as a planar graph. Again, the proof could be presented through whole class discussion.

In all three "proofs" - Königsberg, Euler and planar graphs - it is essential for the student to realise that the result has been conclusively and logically proved, for every possible case, and not just shown to be true for a few specific examples.

The anthropic principle

The impossibility of K_5 leads on to a discussion about the nature of the universe and its dimensionality. After a brief look at how a three-dimensional object would appear to a two-dimensional creature (giving scope for a number of investigations of cross-sections of three-dimensional objects), students then consider how the impossibility of K_5 being planar prohibits the development of intelligent life in a two-dimensional world. Thus the fact that our universe is indeed three-dimensional is another example of the way in which it appears to have been designed for intelligent life, and so can be interpreted as a pointer to the existence of a Creator.

The section ends with opportunity for discussion. Students are asked what features of the world they perceive as being "made-to-measure".

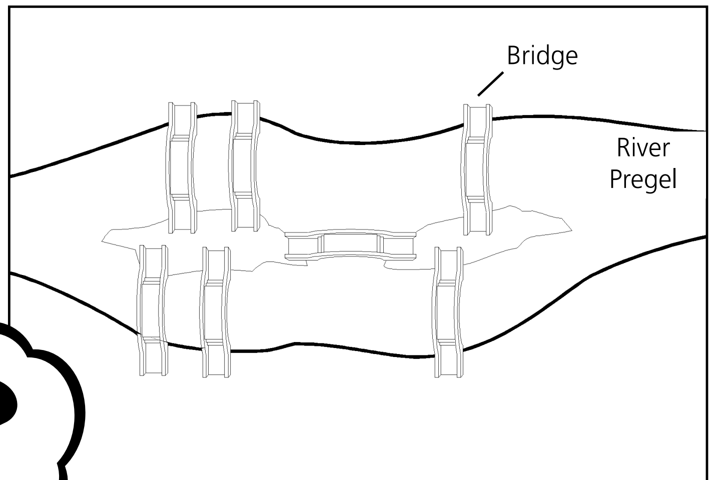
Can you draw it?

UNIT 8

Mathematics is often thought to be about measuring and counting. In this unit we will look at another way of analysing situations, which does not involve measuring distances, but simply looking at connections between points.

The Königsberg Bridges

In 18th Century Prussia there was a town called Königsberg (now called Kaliningrad), through which flowed the River Pregel. There were two islands in the river with various bridges linking the islands to each other and to the banks as illustrated.

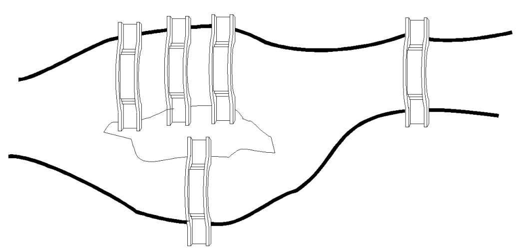


The citizens of Königsberg, out for their Sunday afternoon stroll, tried to solve the following problem: is it possible to walk over every bridge, back to one's starting point, without going over the same bridge twice.

1

1. Try it on paper.
2. If you find it difficult, start with some simpler maps, for example:

a)



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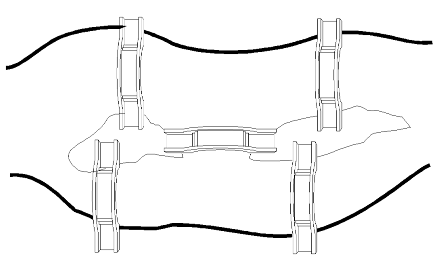

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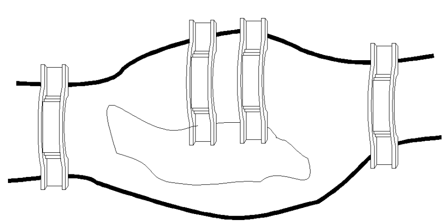
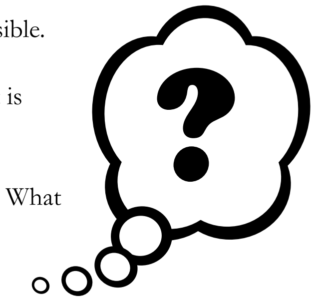
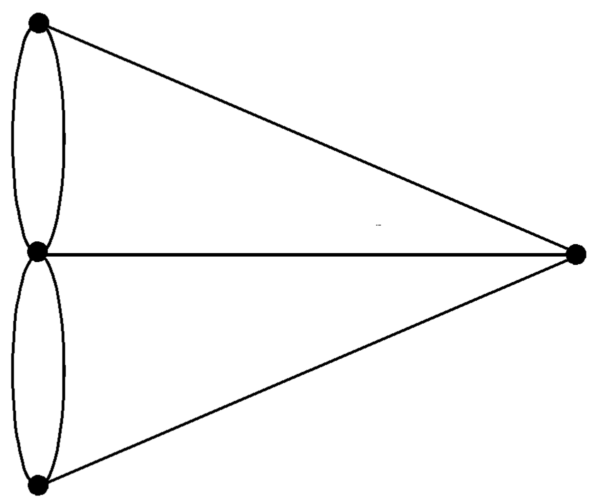




In some cases it is possible - and quite easy, in fact - and in others it may be impossible. In Mathematics, we must always be careful to distinguish between things being impossible and their simply seeming to be impossible. Can we actually prove that it is impossible to walk round the Königsberg Bridges?

The answer is that we can, and you might already have got some idea of a method. What is it that allows some networks, such as 2a) and 2c), to be 'traversable' (i.e., can be walked round without going on the same path twice) whereas others, such as 2b) and the original problem, appear to be 'non-traversable'?

A good start is to re-draw the map in a simpler way. To do this we need to realise that the exact shapes and lengths of the bridges are irrelevant, as are the dimensions of the islands and the banks. What matters is how they are all connected together.



Here the lines represent the bridges, whilst the dots represent the islands and the banks. We have eliminated any detail about the size of the banks or islands, and just highlighted the connections between the islands and the banks. This is what we call a 'topological map': another example is the map of the London Underground, which shows connections but not distances or actual positions. The object is now to trace along every line, until we return to the starting point, without tracing twice over any line.

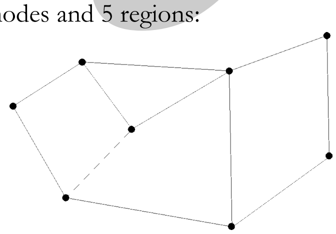
The technical word for a line is *arc*, and for a dot is *node*, and the map is usually called a *network*.
We shall use these words in future.

2

1. Draw networks for 2a), 2 b) and 2c) and try again to traverse them.
2. Draw some networks of your own and see whether they are traversable.
3. In each of the networks you have drawn, count how many of the nodes have an even number of arcs connecting to them (these are called even nodes); count also how many odd nodes there are (i.e. nodes with an odd number of arcs joined to them). Does this help you to decide whether the network is traversable?

Euler's Formula

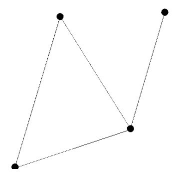
The mathematician who analysed the problem of the Königsberg Bridges and then went on to develop the area of mathematics called topology was Leonhard Euler (1707 - 1783). He went on to establish an important formula in topology, which we will now consider. This formula connects the number of arcs and nodes in a network to the number of separate areas, or regions, produced by the network. For example, this network has 11 arcs, 8 nodes and 5 regions:



(The number of regions includes the area enclosing everything else.)

1. Find the number of arcs, nodes and regions of these networks:

a)



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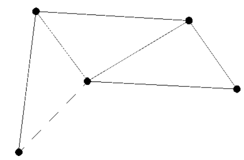

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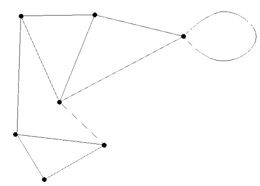

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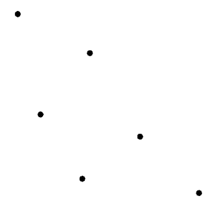


2. Try to find a formula, or equation, relating these three quantities. It may help for you to compile a table with columns N, R and A like this:

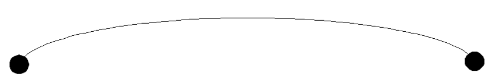
N	R	A
8	5	11

Check your solution with your teacher.

The proof of the result is not too difficult; it works by building up any network from a simple start of 1 node.

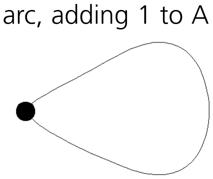
This has one region, around the node, and no arcs, so $A = 0$, $N = 1$ and $R = 1$, so $N + R = A + 2$ in this case. We can now either

i) Add a node, with a connecting arc, increasing A and N by one each, not changing R, and

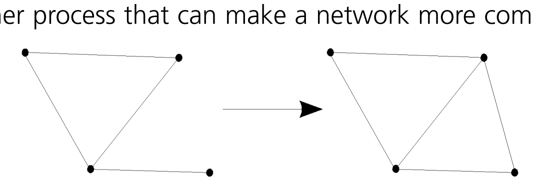


maintaining the original equation; or

ii) Join the node to itself with an arc, adding 1 to A and 1 to R, again maintaining the equation.



For more complicated networks, the only other process that can make a network more complex is joining one node to another, e.g.,

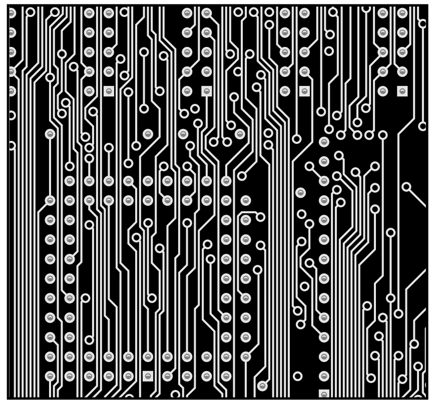


and again this increases A by 1 and R by 1.

So in all cases the equation $N + R = A + 2$ remains true.

Planar graphs

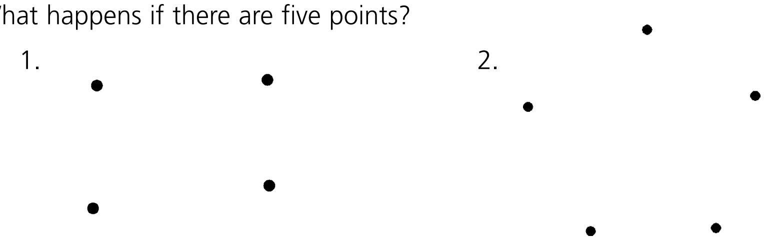
One important modern application of topology is in designing silicon chip micro-processors for computers. These chips consist of a flat layer, with points around the edge which may be connected to external wires, and the purpose of such a design is to produce the correct connections between the points. The connections, of course, must not cross each other, or else there would be a short circuit. Here is a typical design:



These networks with no crossings are called planar graphs, because they can be drawn in a plane.

5

1. If there are just four points, can you join each point to every other point? This graph is called K_4, after the Polish mathematician Kuratowski.
2. What happens if there are five points?



The fact that something seems to be impossible may not mean that it is impossible. Instead, it may just be difficult.

Using Euler's formula, we can actually prove that K_5 is impossible to draw as a planar graph. The proof seems rather odd, in that it starts by assuming that K_5 can be drawn as a planar graph.

If K_5 can be drawn, how many nodes would it have? The answer must be 5. How many arcs would it have? Each of the 5 nodes would have 4 arcs coming out of them which suggests 5 x 4, i.e. 20 arcs. However, each arc will be counted twice because it joins 2 nodes. So there will have to be 10 arcs. Using Euler's Formula ($N + R = A + 2$), how many regions should it have? The answer is 7.

We will now consider the number of regions from a different point of view. Each region must be enclosed by three or more arcs. So, with 7 regions this suggests there will be at least 7 x 3, i.e. 21 arcs. However, since each arc must be the boundary of two regions, there only needs to be $^{21}/_{2}$ i.e. 10.5 or more arcs. Also, because there cannot be a 'half-arc', there must be at least 11 arcs.

We now have what is called a contradiction. Something is wrong, as our argument is suggesting that on the one hand that there are 10 arcs and on the other hand that there are at least 11 arcs.

The reason why we have this problem is not because our step-by-step reasoning is faulty, but because we must have made a wrong assumption at the start. That assumption was that K_5 could be drawn as a planar graph, which we now see leads to nonsense, so we must conclude that in fact it is impossible. (Euler's Formula only applies to planar graphs and so it does not prohibit non-planar graphs.)

6

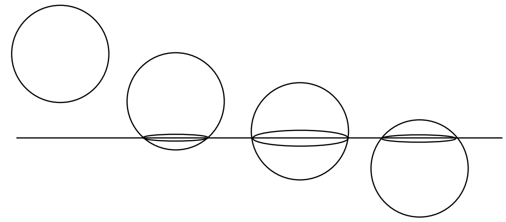
Extra work: Another famous puzzle is to draw $K_{3,3}$ - that is, two sets of three nodes, each joined to all of the other group. This is the gas/water/electricity problem - can you connect all three of the services to three houses, with no pipes or cables crossing over each other?

The anthropic principle

The impossibility of drawing K_5 as a planar graph has an interesting spin-off in theories of the universe.

We all take for granted that the universe has three dimensions of space. Can you imagine a two-dimensional universe where all forms of life live on a flat surface, and there is no third dimension (height)?

How would they perceive a solid object, such as a sphere, passing through their universe?



They would see a circle appear from no-where, gradually increase in radius up to a maximum value, and then shrink away to nothing.

7

1. Draw some diagrams to show how they would see a cube passing through their world.
2. What about a cone passing through?
3. How do you think we would perceive a four-dimensional sphere (a "hyper-sphere") passing through our world?

Is it really possible that there could be intelligent life in a two-dimensional world?

The analysis we have done, showing the impossibility of drawing K_5 in a plane, suggests why our universe has more than two dimensions. Our brains are made of millions of nerve cells (neurones) connected together in a rich and complex interweaving structure. One neurone might be connected to thousands of nearby neurones, passing electrical impulses to and fro. If we lived in a two-dimensional world, such a dense set of interconnections would be impossible because, as we have discovered, it is impossible to join more than 4 items to each other, without crossing and short-circuiting.

Therefore we need at least three dimensions of space for our sort of intelligence to be possible.

There is a similar argument, based on an analysis of planetary orbits, which shows that more than three dimensions would also make the development of intelligence very unlikely. Thus our universe does have just the right number of dimensions to enable intelligent life to exist.

There are all sorts of other remarkable "coincidences" without which intelligent life would be impossible. Examples of these include the following:

- The earth has the right sort of atmosphere;
- It has a protective ozone layer;
- Atoms exist, and are stable, and combine together to form complex molecules;
- The universe is big enough, with the right dynamics, to provide for the formation of galaxies and stars.

8

Can you think of some other ways in which the universe seems to be "made-to-measure"?

The more of these coincidences that scientists discover, the more it seems to some that the universe must have been designed to be inhabited by intelligent life. They call this the 'anthropic principle'.

Possible reactions to this include a) saying that if it wasn't like this then we would not be here to talk about it and b) giving thanks to a Designer.

9

How do you react?