The moment of truth

UNIT 7

This unit allows students to explore ideas associated with truth within mathematics and also to think more broadly about how they come to believe things are true. Most of the activities are within the field of prime numbers and attempt to show that the study of mathematics has a history and that it continues to be an ongoing human endeavour.

Using this unit

In the unit, students investigate a series of statements, conjectures and theorems within the field of prime numbers. In each case, the student has to consider and investigate the truth of what is presented. Following this, there is then scope for discussion of their conclusions and of the different approaches for exploring truth.

Students are also encouraged to reflect on the truth of a wide range of statements and the kind of evidence required to support them.

It is important that the teacher is alert to how students respond to the activities and so reading the teaching notes for "Is this the moment of truth?" is essential before doing the first activity.

The unit requires the students to be familiar with the following:

- AT2 Prime numbers, factors and how to test for them;
  Powers and index notation.
- AT3 Using formulae given in symbolic notation.

The unit's length depends on time given for investigating each statement, but requires a minimum of two hours. The work is suitable for Intermediate/Higher level students.

As well as the main unit sheet, the students will require the "Prime 2000" sheet and the statement cards. It is suggested to students that they might find that a computer supports their investigation. Either a spreadsheet or simple programming would be of value.

Mathematical content

AT1
To develop skills of mathematical reasoning and consider different lines of mathematical argument, in particular:

- To make and test conjectures
- To analyse results to see if they are valid
- To understand general statements
- To appreciate the difference between mathematical explanation and experimental evidence
- To use mathematical reasoning when following a line of argument and appreciate different ideas of proof.

Spiritual and moral development

The aim of this unit is to encourage students to develop a respect for truth in all its forms. It helps them to see that the quest for truth applies in all fields of human discovery. It also aims to encourage them to reflect about their own ideas of truth and how they are formed.

Background

The National Curriculum correctly identifies mathematical reasoning as an important part of using and applying mathematics. It needs to be questioned, though, whether students are really receiving opportunities to develop in this area. Have most students instead developed only the skills of naive empiricism? Is it fair to say then that students are "accepting truths for the wrong reasons"? As mathematics teachers, it is our responsibility to show students that there is such a thing as deductive proof while at the same time helping them to appreciate the limitations of mathematics.

Mathematics is also a human activity that has developed in many different cultures throughout history. However, students often have no notion of how the mathematics they study in a GCSE textbook was originally developed or for what purpose. By offering students a chance to encounter the history of mathematics and biographies of mathematicians we may then expand their view of the nature of mathematics.

Additional sources

N. Balacheff, "Aspects of proof in pupils' practice of school mathematics" in *Mathematics, Teachers and Children* (OU/Hodder and Stoughton, 1988)
E. T. Bell, *Men of Mathematics* (Victor Gollancz Ltd., 1937).
D. M. Burton, *The History of Mathematics, An Introduction* (Allyn and Bacon, 1985).
M. Kline, *The Loss of Certainty* (O.U.P., 1980).
K. Porteus, "When a Truth is Seen to be Necessary", *Mathematics in School, 23.5* (1994) p. 2-5.

Notes on the activities

To help students make links between investigating truth in mathematics and truth in other fields, it is worth beginning with the *Nothing but the truth?* activity (see page 4 of teacher's notes) and then returning to it again after the prime number work.It may then be necessary to review with students their understanding of prime numbers and how to check for them. This is a good time to give them the "Prime 2000" sheet.

Is it true?

This activity sets the scene for the discussion and reflection that follows in *Is this the moment of truth?* and *Nothing but the truth?* Students should be given the general work sheet and as many statement/ background cards as is desirable. (The 'statements' page should be photocopied with the 'backgrounds' page on the reverse before cutting up the statements.) The activity would be best done in groups working independently of each other and only reporting results during the discussion in *Is this the moment of truth?* When the students are investigating the different statements it is important to encourage them to think about how the truth could be justified or checked even if they cannot do it for themselves.

Statement 1:

a) Prime (on "Prime 2000" sheet)
b) Prime (on "Prime 2000" sheet)
c) Not prime since $48253 = 73 \times 661$
d) Prime. Mersenne prime: $2^{19} - 1$
e) Prime. Mersenne prime: $2^{127} - 1$. Verified by Lucas in 1876 and remained as world record holder until 1951.

Statement 2:

An ideal opportunity for the use of IT. True for all values of N to 40, but it inevitably breaks down for $N = 41$. When $N = 41$, $N^2 - N + 41 = 1681$. It then produces both prime and composite values.

Statement 3:
This is the famous Goldbach conjecture, which at the time of writing is still believed to be true, but as yet not proved.
Euler, in his reply to Goldbach, wrote, "I consider it to be a theorem which is quite true, although I cannot demonstrate it".

The conjecture has been thoroughly checked by computers and the results suggest that it is likely to be true. However, since no proof has been found we still do not know. Indeed, Goldbach's conjecture may fall into the category suggested by Godel of being a true proposition, but impossible to prove.

Statement 4:
This is true. It was in fact proved by Euclid. It is worth trying to give students a flavour of the proof so that they appreciate the place of formal proof:

Assume the number of primes is not infinite and there is a last prime, p.

Consider the number, $N = 2 \times 3 \times 5 \times 7 \times \ldots \times p$, the product of all the primes in their order from 2 to p.

The number $N + 1$ is not divisible by any of these primes as there is always a remainder of 1. So the number is either:

a) Composite, but divisible by a prime larger than p;

or

b) Prime and obviously also larger than p.

In either case, we have a contradiction to our original assumption that the primes are finite, so we conclude they are indeed infinite. (It may help to consider two cases such as $p = 7$ when $N + 1$ is 211 (prime) and $p = 13$ when $N + 1$ is 30031 (composite) since $30031 = 59 \times 509$.

Statement 5:
These are now known as the Mersenne primes, although others had done a lot of work on them. For N up to 19, the primes were accepted already in 1644. $N = 31$ was not verified until 1750 by Euler. $N = 127$ was verified by Lucas in 1876. In 1883 it was shown that $N = 67$ did not work, but $N = 61$ did. Was it a misprint in Mersenne's work? In the period 1911-1914, $N = 89$ and $N = 107$ were found to work and later $N = 257$ was shown not to.

Statement 6:
Like statement 2, this is a very successful formula and works for $N = 1$ to 79.
It breaks down at 80, but only fails for four other values before 100: 81, 84, 89, 96.

Statement 7:
These are known as the Fermat numbers.
For $M = 1, 2, 4, 8, 16$ the numbers are: 3, 5, 17, 257, 65537. They are all prime.

However, in 1732, Euler showed that for $M = 32$ the number is 4294967297 and that this is composite since it equals 641×6700417. Also, no other Fermat number has been found to be prime, so that the opposite is now claimed, that for powers of two greater than power 4, Fermat numbers are composite.

Statement 8:
Like Goldbach's conjecture, this has been well checked by computers, but no proof has been found.

Statement 9:
Students should be encouraged to come up with some conjecture. They might try adapting the Goldbach or Levy conjectures.

Is this the moment of truth?

Throughout *Is it true?* students will have been discussing ideas of truth and verification. The purpose of this stage is to draw those thoughts together. This could be done all at once or the "moment of truth" for different statements could be spread over a number of lessons.

Below are suggested discussion points and the statements to which they might be particularly linked.

a) We believe many things to be true because we trust the source of the truth, either people or documents.

For example, trusting the "Prime 2000" sheet, trusting other people who have worked on these statements before us and trusting the teacher throughout the "moment of truth" stage.

b) We sometimes want to believe things are true and so do not go out of our way to check them out.

For example, the formulas in statements 2 and 6 work for a long time, so we do not really want them to break down.

c) We can sincerely believe something to be true, but that cannot in itself make it true.

For example, Fermat and Mersenne may have sincerely believed their statements to be true.

d) We can use computers to check out the truth of statements, but how reliable are they, how much trust do we put in them?

For example, in Statements 2 and 6 we may have used computers, but have they led us to the truth? Why do we think so?

e) It is possible to prove some things are true by logical argument, but this always depends on accepting an earlier truth or axiom.

For example, statement 4 is provable, but the argument uses previously established results.

f) We can believe things are true because in our experience they are always true, but this is not always enough.

For example, statements 2, 5, 6 and 7 work for some of the time but break down.

g) It is possible for things to be true even if we haven't proven them to be true and also for things to be true even though it may be impossible to prove they are true. If this is the case in mathematics then it is bound to apply elsewhere as well.

For example, statements 3 and 8 could be true but not provable.

h) It is worth searching for truth, for in doing so we will often also find new truth.

For example, in working on statements 3, 5 and 7 mathematicians have discovered other results and developed new mathematical methods.

Nothing but the truth?

This activity is best done individually, initially, so that students can respond personally. It could be set as a homework task. Hopefully, pupils will realise that some of the ideas about truth discussed in *Is this the moment of truth?* have a broader application than just the field of mathematics.

Euler biography (extension activity)

Much of the unit has drawn the students' attention to the idea of mathematics having a history. It is suggested then that the students could read a biography of Euler in order to appreciate more about a particular mathematician working in this field. Using a biography of a mathematician and reflecting on the questions suggested below will be an unfamiliar activity for most students. They will need encouragement to think for themselves and to express their own views. It could be that the activity once clarified might also become a homework task.

Questions to work on after reading the biography:

- Why do you think Euler studied mathematics?
- Why do you study mathematics? What is its value from your point of view?
- Euler was a committed Christian. Do you think it is possible to prove whether God exists or not?

The moment of truth

UNIT 7

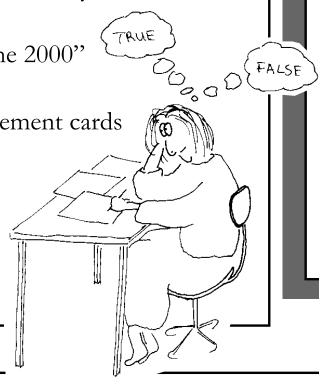
If you have ever had an experience where you have finally found out whether something is true, then you will know what is meant by "the moment of truth". This unit will have many such moments as you investigate statements that have been made by people in the past.

You will learn about how you can test to see if things are true in maths and how you can show why you believe something to be true or false.

You will be mainly working with prime numbers and will also need to use simple formulas, so tell your teacher if you need any help in those areas before you start.

As well as this sheet, you also need:

a) The "Prime 2000" sheet
b) The 9 statement cards



1 Is it true?

Statement cards 1-8

For each statement that you work on do the following:
a) Read the statement and check you understand what it means;
b) Write down your "First Reaction" - true or false;
c) Read the background information that is given;
d) Investigate to see if the statement is true or not and record any work you do;
e) Write down your "Final Conclusion" - true or false and give your reasons for this conclusion. (However, if you are still not sure about the statement after investigating then suggest how someone might be able to check it out).

Statement card 9

This is your chance to become famous! Having now done quite a lot of work on prime numbers, perhaps you can produce a statement. Remember that statement 7 was suggested by someone who was not a full time mathematician but enjoyed exploring mathematics for himself.

When you think you have devised a statement, first check it out yourself and then try it out on someone else.

2 Is this the moment of truth?

In this second activity you will have to explain your "Final Conclusions" to other students and be prepared to justify your belief. Each statement will be discussed in turn and different arguments can be put forward.

When the discussion is complete, it will be time for "The Moment Of Truth".
Your teacher will reveal the truth about the statement.

Nothing but the truth?

In this unit, you have been investigating whether things are true or not within mathematics and how you can know. In this activity you will consider statements from other areas as well.

For each statement work through the following stages:

a) Read the statement and think about it;
b) Write down what you think: true, false or don't know;
c) Write down the kind of evidence you would want to convince you.

1. The angles in triangles always add up to 180 degrees.
2. The world is flat.
3. Henry the Eighth became King in 1509.
4. Smoking causes cancer.
5. Jesus Christ rose from the dead.
6. Human beings evolved from apes.
7. There is no life after death.
8. The world will end in 1999.
9. There is no life on the moon.
10. Mathematics teachers are on average more intelligent than English teachers.

Leonhard Euler (1707-1783)

Leonhard Euler (pronounced Oiler) was born in Basle, Switzerland on April 15, 1707. He became the most prolific mathematician there has ever been. His life works are estimated to need 70 huge volumes, if they were to be written up in full.

His father, Paul, was a church pastor and also an accomplished mathematician, having been a pupil of Jacob Bernoulli. Paul intended his son to become a church minister, but he made the mistake of teaching him mathematics. Leonhard was hooked. However, he went to the University of Basle to study theology.

While there, he became friendly with various members of the mathematical family, the Bernoullis. They soon recognised his mathematical talent and persuaded his father that Leonhard was destined to become a great mathematician. He had hoped to become a professor at Basle, but in 1727 he left Switzerland for Russia to work at the academy in St Petersburg. By 1733, at the age of 26 he was the leading mathematician there.

While in St Petersburg he married Catharina. Together they had 13 children, but only five survived beyond a very young age. He was one of the great mathematicians who could work in any condition. It is said that he would happily work with his children around him and a baby on his lap. It was during his first period in Russia that he lost the sight of his right eye following an illness possibly brought on by overwork.

In 1741, he moved to Berlin where he remained for 25 years at the academy. Here he did some of his finest work. However, he did not get on well with Frederick the Great and so when an invitation came from Catherine the Great in Russia to return to St. Petersburg, he felt this was the best thing for his family to do. It was at this time that he began to lose the sight in his other eye and it was not long before he was totally blind. This did not affect his rate of output, in fact he commented, "I'll have fewer distractions".

He had an amazing memory and his speed of calculation was phenomenal. He could do calculations in his head that other mathematicians had difficulty doing on paper. His reputation for this amazing ability was such that when he died it was said, "He ceased to calculate and to live".

Euler remained a committed Christian throughout his life, often leading his family in prayers and reading the Bible to his children. His Christian beliefs assured him that God had entrusted people with the task of understanding God's laws. He believed that God had made a perfect universe and that with persistence we could learn much about it.

Prime 2000

2	127	283	467	661	877	1087	1297	1523	1741	1993
3	131	293	479	673	881	1091	1301	1531	1747	1997
5	137	307	487	677	883	1093	1303	1543	1753	1999
7	139	311	491	683	887	1097	1307	1549	1759	
11	149	313	499	691	907	1103	1319	1553	1777	
13	151	317	503	701	911	1109	1321	1559	1783	
17	157	331	509	709	919	1117	1327	1567	1787	
19	163	337	521	719	929	1123	1361	1571	1789	
23	167	347	523	727	937	1129	1367	1579	1801	
29	173	349	541	733	941	1151	1373	1583	1811	
31	179	353	547	739	947	1153	1381	1597	1823	
37	181	359	557	743	953	1163	1399	1601	1831	
41	191	367	563	751	967	1171	1409	1607	1847	
43	193	373	569	757	971	1181	1423	1609	1861	
47	197	379	571	761	977	1187	1427	1613	1867	
53	199	383	577	769	983	1193	1429	1619	1871	
59	211	389	587	773	991	1201	1433	1621	1873	
61	223	397	593	787	997	1213	1439	1627	1877	
67	227	401	599	797	1009	1217	1447	1637	1879	
71	229	409	601	809	1013	1223	1451	1657	1889	
73	233	419	607	811	1019	1229	1453	1663	1901	
79	239	421	613	821	1021	1231	1459	1667	1907	
83	241	431	617	823	1031	1237	1471	1669	1913	
89	251	433	619	827	1033	1249	1481	1693	1931	
97	257	439	631	829	1039	1259	1483	1697	1933	
101	263	443	641	839	1049	1277	1487	1699	1949	
103	269	449	643	853	1051	1279	1489	1709	1951	
107	271	457	647	857	1061	1283	1493	1721	1973	
109	277	461	653	859	1063	1289	1499	1723	1979	
113	281	463	659	863	1069	1291	1511	1733	1987	