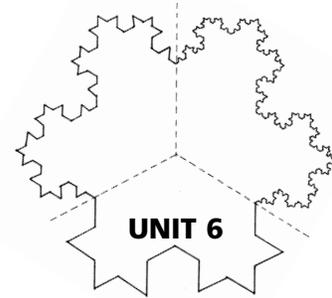


# Fractals



In this unit, students will learn about fractals, mainly through the construction of fractal images. At the same time, it is hoped that students will find fractals fascinating and will appreciate that the created world, which fractals only model in a limited way, is even more complex and awe-inspiring.

## Using this unit

It is not intended that students attempt every activity in the unit.

**There are two main constructions**, the Koch Snowflake and the Sierpinski Triangle. Neither are difficult to construct but a reasonable degree of accuracy is required and some patience in the latter stages! Diagrams, in various stages of development, are included for reference. The construction of the Koch Snowflake to the number of generations suggested will take about 45 minutes. The Sierpinski Triangle takes less time but colouring it is almost irresistible! It is the best activity to choose for a group which finds accurate construction difficult.

**There is also a section containing some challenging algebra** in order to calculate fractal dimensions. Students working at level 8 or above should be able to tackle the problem, but the section is entirely optional and can be omitted. Extra time will be needed if the algebra is attempted.

**There are then three ideas for further work.** The Square Fractal is similar in construction to the Snowflake but is more complicated to draw. Again a diagram is enclosed for reference. The algebra is similar to that for the Koch Snowflake and is again optional.

A wall display on fractals is likely to create interest beyond the maths classroom. Finally, there are close links between Pascal's Triangle and the Sierpinski Triangle which are interesting to explore.

This unit explores areas of mathematics which are unfamiliar to most students, but the activities are likely to be suitable for pupils working at Intermediate/Higher levels. Most of the activities require accuracy in drawing and to get the best effects students must be prepared to construct very accurately. Some sections require a reasonably high level of reading ability and could be read to the class.

❖ Students will need plain A4 paper; a sharp pencil; a rubber; a ruler and a pair of compasses. A calculator is needed for the optional algebra section.

## Mathematical content

### AT2

- ◆ Ratios
- ◆ Rearranging formulas involving indices (optional)
- ◆ Trial and improvement methods for solving exponential equations

### AT3

- ◆ Perimeters
- ◆ Constructions and accurate drawing

## Spiritual and moral development

It is hoped that pupils will enjoy the fascination of fractal curves; that they will be awed by the complexity of the created universe and that they will appreciate that mathematics, far from excluding God, can be an introduction to Him.

## Background

Students tend to view mathematics as a very cut and dried subject with no unanswered questions. Fractals offer them a chance to take a look at some mathematics which has only been studied during this century and is still being developed.

The world of fractals is a fascinating one. Fractals can model the natural world and can produce images of startling complexity and beauty. Mathematicians have frequently attempted to describe the world we live in using mathematical models. Many felt, as Kepler did, that they were “thinking God’s thoughts after Him” and modern mathematicians are no less intrigued by the idea that they are discovering more and more of the wonders of creation. The ultimate fractal, the Mandelbrot set, has been described as the ‘thumb print of God’ by a mathematician working in this field. Mandelbrot himself, who pioneered this work on fractals, is fascinated by the possibility of describing natural phenomena with fractal equations and never seems to lose his sense of wonder at the world he is exploring. He does not claim to have invented anything, simply to be discovering what has already been created.

The mathematics used to generate the amazing fractal patterns produced on computers involves the iteration of simple formulas, which, although not complicated, do require a knowledge of complex numbers. No attempt is made to investigate this here but the geometric idea of a simple pattern for ever repeating itself can be conveyed effectively through drawing activities.

## Additional sources

### Books:

- I. Stewart, *Does God Play Dice? The New Mathematics of Chaos* (Penguin, 1990).
  - M. E. Lines, *Think of a Number* (Adam Hilger, 1990).
  - McGuire, *An Eye for Fractals* (Addison Wesley, 1991).
  - R. Dixon, *Mathographics* (Blackwell, 1987).
  - D. Uribe, *Fractal Cuts: Exploring the Magic of Fractals with Pop-up Designs* (Tarquin, 1993).
- This book consists of wonderful fractal pop-up cards to make.

### Video:

Prism Leisure Video, *The Colours of Infinity* presented by Arthur C. Clarke available from:  
New Moon Productions, Unit 1, Baird Road, Enfield, EN1 1SQ.

### Computer package:

The famous freeware package, *Fractint* (motto: “Don’t want money. Got money. Want admiration.”) is available on the Internet. It is produced by the Stone Soup Group.

### CD-ROM

*Exploring Chaos and Fractals* (Royal Melbourne Institute of Technology) available in PC format from  
Microinfo Ltd., Omega Park, Alton, Hants., GU34 2PG (Tel: 01420 86848).

Fractal postcards are available from:

The Mathematical Association, 259 London Road, Leicester, LE2 3BE.

## Notes on the activities

### The Koch Snowflake

A diagram is enclosed showing this curve in three stages of construction. It does become very time consuming in the latter stages but many pupils will be fascinated and will want to carry on as far as possible. It should soon become clear why fractals were never really explored until the birth of the computer made the many repetitions possible. The book “Mathographics” referred to in the additional sources gives a simple programme in BASIC which can be used to draw the snowflake on a computer.

The fascination of the snowflake is that the curve is infinite but the area it encloses is finite. This idea is explored in the student worksheet.

#### Task 2. Examples of possible answers are:

1. A straight line
2. A circle
3. Triangles can be added “for ever” and so the curve is of infinite length.
4. A circle can be drawn so that the snowflake is inscribed within it. The area inside the circle is finite. The area inside the snowflake is smaller than that inside the circle and, therefore, is also finite.

### Fractal dimensions

The fractal dimension measures the degree of ‘crinkliness’ of the curve, how well it fills the space. A smooth curve would have a dimension of 1.

The algebra of fractal dimensions can be omitted but it is an interesting challenge for pupils who can use negative indices to rearrange Mandelbrot’s formula into the given form which is easily solved using trial and improvement on the calculator.

#### Students should arrive at the fractions:

$$\frac{L_{n+1}}{L_n} = \frac{1}{3} \quad \text{and} \quad \frac{P_{n+1}}{P_n} = \frac{4}{3}$$

Substituting in Mandelbrot’s formula gives;

$$\frac{4}{3} = \left(\frac{1}{3}\right)^{1-D}$$

$$4 = 3 \times (3^{-1})^{1-D}$$

$$4 = 3 \times 3^{-1} \times 3^D$$

$$4 = 3^D$$

This can easily be solved using logarithms, hence;

$$D = \frac{\log 4}{\log 3} = 1.26 \text{ to 2 decimal places}$$

### The Sierpinski Triangle

This is a satisfying construction which demonstrates self similarity very clearly. It is very effective when coloured. It is different from the Snowflake construction in that, if continued to infinity, the original triangle will be filled with holes, and it will disappear to fractal dust. Again, a diagram is included for reference.

### The square fractal (reinforcement activity)

This is harder to draw as the corners continually meet. Again, a drawing in three stages of development is included. Shading the ‘inside’ makes the true shape clear. It quickly becomes obvious that when continued ‘to infinity’ the curve would enclose a square containing tiny dots of space, fractal dust. The algebra is very similar to the Snowflake except that the ratio of successive perimeters is 5:3 and so the resulting equation is  $3^D = 5$  with solution  $D = 1.46$  to 2 decimal places.

### The wall display (extension activity)

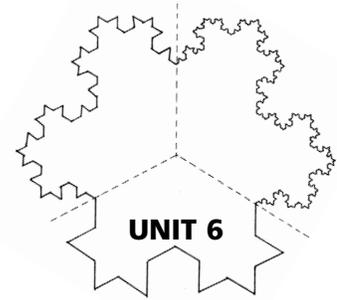
The designs produced by the students using this unit make a striking display. There are many natural objects exhibiting self similarity and pictures that bring out this property could be added to the display. Suggestions are: the edges of cumulus clouds; fir trees and ferns; the rocky ridges of mountains; aerial views of rivers and their tributaries; trees in winter etc.. There is also a growing supply of commercially produced material. Some are listed on the previous page.

### Pascal’s Triangle (extension activity)

A supply of photocopies of Pascal’s Triangle (at least 15 lines) would be useful for this investigation. The numbers are best enclosed in boxes so that the patterns are clear. Shading the odd numbers shows the same pattern as the Sierpinski Triangle. Other fractal patterns can be generated by leaving unshaded all multiples of a particular number and shading every other number.



# Fractals



Have you ever wondered what mathematicians do these days? Perhaps you think that this is a cut and dried subject where all the loose ends were tied up long ago, if not with Pythagoras then certainly by the time of Newton! If so, it will surprise you to learn that there is a great deal of exciting research taking place in mathematics. So what are they doing? This unit introduces you to some of the ideas which are intriguing mathematicians today.

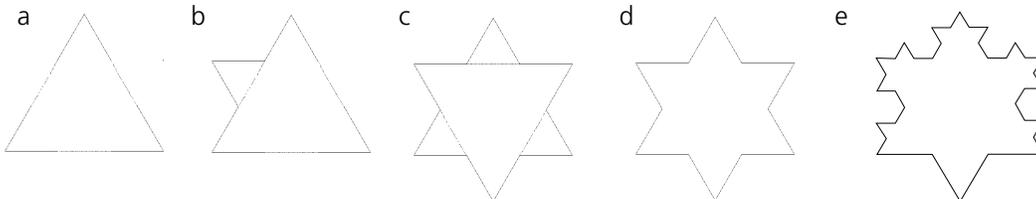
You will need a sharp pencil, a ruler, a rubber and compasses to construct an equilateral triangle. To calculate the fractal dimensions you need to be able to solve equations using a trial and improvement method on your calculator.

**'Fractals capture  
the texture of  
reality'**  
Jeanne  
McDermott

## The Koch Snowflake



Take a piece of plain A4 paper and construct an accurate equilateral triangle in the middle, making each side 9cm long. Divide each side into three equal parts, (see diagram a).

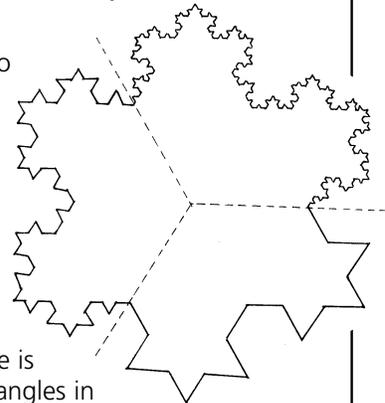


On the centre 'third' of one side, construct an equilateral triangle of side 3cm, (diagram b). Repeat on the other two sides. Your diagram should now look like the 'star of David', (diagram c).

Rub out any construction lines and also rub out the interior lines so that only the outline of your shape is left, (diagram d).

On each side of each point of the star, repeat the above process so that you are constructing 1 cm equilateral triangles on the middle 'third' of each 3cm line segment. If you are very careful, it is possible to do this using a compass construction. Again rub out your construction lines and the interior lines. The diagram e shows this stage partially completed.

Repeat the process one more time, although the new triangles are too small to construct accurately. However if you measure about 3mm in from each end of each line segment it is easy enough to guess the position of the top of the triangle. This stage is fiddly and time consuming as you must place one of these tiny triangles in the centre of each 1cm line segment, but it gives you a good impression of the snowflake curve.



In theory, we could go on repeating this process for ever and the resulting shape is called the Koch Snowflake curve. It was first drawn by the German Mathematician Helge von Koch in 1904 and its fascination lies not just in its resemblance to a snowflake, but in the fact that it is a curve with an infinite length but it encloses a finite area. This is an unusual combination.

Many curves have infinite length and others enclose a finite area, but these two properties do not usually occur together.



1. Describe a curve of infinite length which does not enclose a finite area.
2. Describe a curve of finite length which does enclose a finite area.
3. Explain why the Koch Snowflake has a perimeter of infinite length.
4. Draw a simple geometric shape around the Koch Snowflake. How does this demonstrate that the Koch Snowflake encloses a finite area?

This curve is an example of a fractal. A fractal is a curve of infinite length, (in our example, we could, in theory, have gone on adding triangles for ever). What is more, however much we magnify the edge of the curve, we will still see the same shape. This property of fractals is called 'self-similarity' and it is a property which makes fractals both beautiful and useful.

Many natural phenomena exhibit self-similarity.

One of the first questions asked in this field of mathematics was "How long is the coastline of Britain?"

**Can you see the connection?**

If we look at the coastline on a map and measure it with a bit of string, we will get a very different answer from the one we would get if we set off to scramble along the shoreline. The coastline is crinkly, but strangely enough the crinkly shape you see on a map is very similar (although on a very different scale) to the shape you see if you examine a rock jutting out to sea. Coastlines exhibit self-similarity. So too do clouds, trees and mountain ranges.



Until fractals were discovered, geometry had not been able to cope very well with such shapes.

Try to describe a tree using only circles, rectangles and triangles!

Fractals offer Mathematicians a simple way of mimicking these natural shapes very accurately.

Computers can take a very small piece of information and (because of the self-similarity of the object) they can build up a fractal picture which looks remarkably like the real thing.

If you have seen the films 'Star Trek II' or 'Return of the Jedi' you will have seen these 'fractal forgeries'. They were used to create the planet landscapes.

**"The book of nature is written in mathematical language"**

Galileo

### Fractal dimensions

Apart from followers of 'Dr. Who', most people believe there are three dimensions, but fractals seem to fall in between dimensions!



Return to your snowflake curve. When you drew the first triangle you had an edge length of 9cm and a perimeter of 27cm. Let us call these

$$L_0 = 9 \quad \text{and} \quad P_0 = 27$$

Now consider the first stage of the construction, the 'star of David'. How long is the length of a side now?

$$L_1 = 3. \quad \text{How long is the new perimeter } P_1?$$

Complete the following table of results for your snowflake curve:

	Side length (L)	Perimeter length (P)
9cm triangle	$L_0 = 9$	$P_0 = 27$
3cm triangle	$L_1 = 3$	$P_1 =$
1cm triangle	$L_2 = 1$	$P_2 =$
$\frac{1}{3}$ cm triangle	$L_3 = \frac{1}{3}$	$P_3 =$

Now investigate the fractions

$$\frac{L_3}{L_2} \quad \frac{L_2}{L_1} \quad \text{and} \quad \frac{L_1}{L_0}$$

Complete the statement

$$\frac{L_{n+1}}{L_n} =$$

Now do the same thing for your perimeters and complete the statement

$$\frac{P_{n+1}}{P_n} =$$

The value you obtained for  $D$  for the snowflake curve is a measure of its roughness or crinkliness. The dimension of a coastline is about 1.2, so a coastline is about as crinkly as the Koch snowflake!

Benoit Mandelbrot was born in Warsaw in 1924 and pioneered mathematical research in this field. In fact Mandelbrot first came up with the name fractals and had a particularly important fractal named after him. Mandelbrot defined the dimension of a fractal, a measure of its crinkliness! The dimension,  $D$ , of a fractal is found from Mandelbrot's equation

$$\frac{P_{n+1}}{P_n} = \left( \frac{L_{n+1}}{L_n} \right)^{1-D}$$

You should be able to put your answers for the snowflake into this equation to give:

$$4/3 = (1/3)^{1-D}$$

This is a difficult equation to solve without some A level mathematics, but it can be rearranged into the form:

$$3^D = 4$$

You may be able to rearrange it yourself if you have studied negative indices.



With the equation in this form, you can solve it by a trial and improvement method on your calculator. The first three steps are done for you.

Try  $D = 1$        $3^1 = 3$       too small

Try  $D = 2$        $3^2 = 9$       too big

Try  $D = 1.5$        $3^{1.5} = 5.19615242$       too big

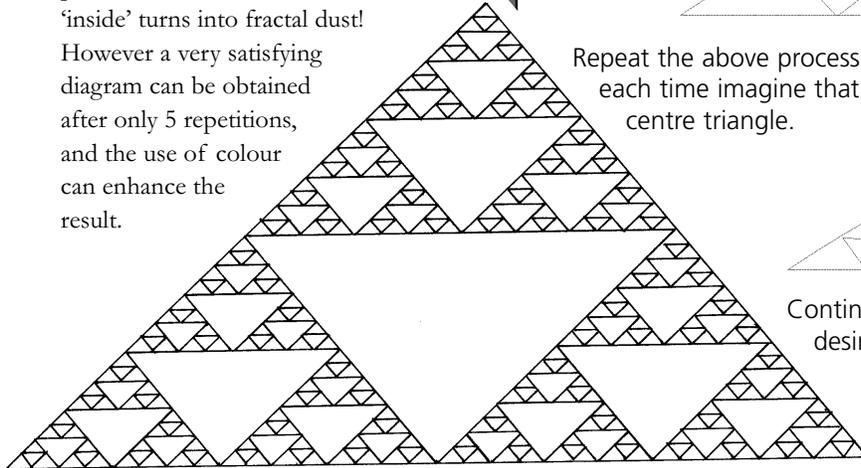
Try  $D = 1.4$        $3^{1.4} =$

Continue to search until you can give the value of  $D$  to two decimal places.

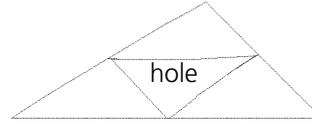
### The Sierpinski Triangle

The Koch snowflake was interesting because it had an infinite perimeter but a finite area. When you constructed it, you built it up from the inside. Our next fractal starts with an outline and removes sections from inside at each stage. If we continue the process to infinity we end up with an infinite perimeter and no area at all. The 'inside' turns into fractal dust!

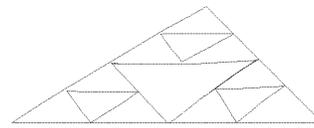
However a very satisfying diagram can be obtained after only 5 repetitions, and the use of colour can enhance the result.



On plain A4 paper draw a large scalene triangle. Mark the mid-points of each side and join them together. Imagine that the new triangle which is formed is removed from the diagram leaving three triangles around the 'hole'.



Repeat the above process for the three triangles, each time imagine that you have removed the centre triangle.



Continue the process as far as desired.

#### Fractals have enormous fascination.

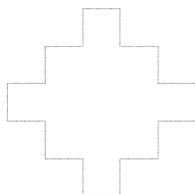
Computers can create fractal patterns by using colours to highlight different areas. The results are extremely beautiful. Fractal geometry allows us to describe mathematically many shapes we see in the natural world much more accurately than with conventional geometry. A fractal tree with dimension 0.6309 is remarkably like the real thing!

Mathematicians struggle to describe more and more accurately the intricacy, beauty and order of the world around us. Those working in this area do not feel they are inventing, they feel they are discovering something that is already there. One described the Mandelbrot set, an important and basic fractal named after Mandelbrot, as the 'thumb print of God'. Mathematics does not lead us away from the idea of a Creator but can lead us towards one.

#### The square fractal (extension activity)



Start with a 9cm square in the centre of A4 paper. As before, divide each side into three equal parts and construct a 3 cm square on each centre third, removing the internal lines.



Calculate  $P_0$  and  $P_1$ ,  $L_0$  and  $L_1$ , for this shape and calculate the fractions

$$\frac{P_1}{P_0} \quad \text{and} \quad \frac{L_0}{L_1}$$

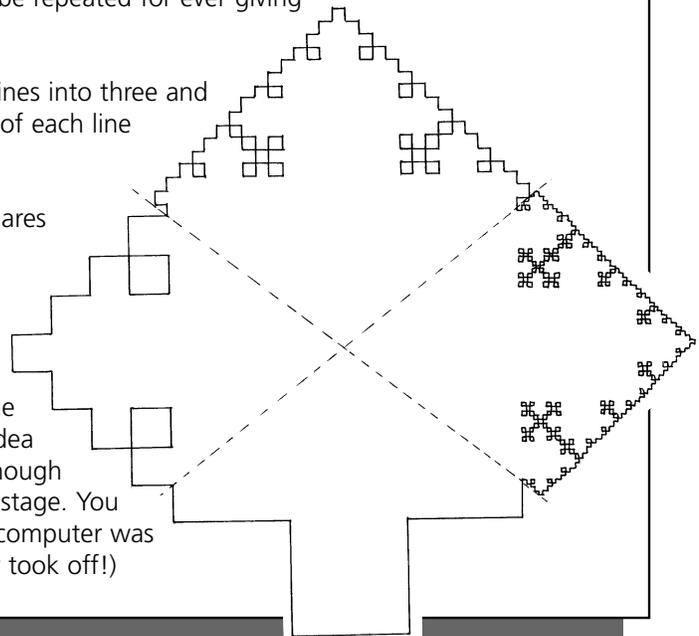
(Activity 6, The square fractal, continued)

As with the snowflake the process can be repeated for ever giving a fractal curve.

The next stage is to divide all the 3cm lines into three and place a 1cm square on the centre third of each line segment.

Unfortunately you will find that the squares begin to touch at the corners making it more difficult to see which bits are 'outsides' and which bits are 'insides'!

Once you have drawn all the 1cm squares you should be able to colour the 'inside' of your shape and get a good idea of the general shape. (It is possible, although very time consuming, to draw the next stage. You will soon see why the invention of the computer was necessary before fractal geometry really took off!)



What outline shape do you think this fractal will fill if you continue the process indefinitely?

What will happen to the internal holes?

To calculate the dimension of the 'square' curve use Mandelbrot's equation as you did before but with the fractions you have just found. This time you should get

$$5/3 = (1/3)^{1-D}$$

which rearranges to

$$3^D = 5$$

Using a trial and improvement method, solve this equation to 2 decimal places and so state the dimension of the 'square' curve.

### Wall display (extension activity)



Produce a wall display using your drawings.

Look for books in the library which deal with fractals. The Mathematics might look complicated but the pictures are stunning. Fractal pictures are now being used for a variety of posters, cards, etc.. Keep a look out for them.

Which natural shapes demonstrate self-similarity?

Make a collection of pictures and add them to your wall display.



### Pascal's Triangle (extension activity)



If you have studied Pascal's Triangle you will find some remarkable similarities between the number patterns it contains and the geometric patterns in the Sierpinski Triangle.

Investigate.